Navier-Stokes equations with periodic boundary conditions and pressure loss

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Abstract

We present in this note the existence and uniqueness results for the Stokes and Navier-Stokes equations which model the laminar flow of an incompressible fluid inside a two-dimensional channel of periodic sections. The data of the pressure loss coefficient enables us to establish a relation on the pressure and to thus formulate an equivalent problem.

Keywords : Navier-Stokes equations, incompressible fluid, bidimensional channel, periodic boundary conditions, pressure loss.

1 Introduction

The problem which one proposes to study here is that modelling a laminar flow inside a two-dimensional plane channel with periodic section. Let Ω be an open bounded connected lipschtzian of $\mathbb{R}^2$ (see figure hereafter), where $\Gamma_0 = \{0\} \times \left]-1,1\right[$ and $\Gamma_1 = \{1\} \times \left]-1,1\right[$.

One defines the space

$$V = \left\{ \boldsymbol{v} \in \mathbf{H}^1(\Omega) ; \operatorname{div} \boldsymbol{v} = 0, \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_2,\ \boldsymbol{v}_{|\Gamma_0} = \boldsymbol{v}_{|\Gamma_1} \right\}$$

and for $\lambda \in \mathbb{R}$ given, one considers the problem

$$(\mathcal{S}) \begin{cases} \text{Find } \boldsymbol{u} \in V, \text{ such that} \\ \forall \boldsymbol{v} \in V,\ \displaystyle\int_\Omega \nabla \boldsymbol{u} . \nabla \boldsymbol{v}\ d\boldsymbol{x} = \lambda \int_{-1}^{+1} v_1(1,y)\ dy. \end{cases}$$

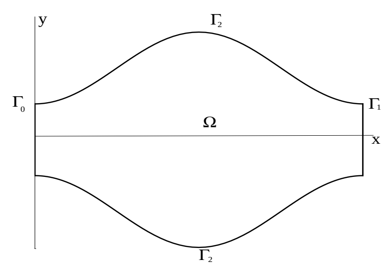

Figure 1. Geometry of channel

2 Resolution of the problem $(\mathcal{S})$

Initially one proposes to study the problem $(\mathcal{P})$. One has it

Theorem 2.1 *Problem* $(\mathcal{S})$ *has an unique solution* $\boldsymbol{u} \in V$. *Moreover, there is a constant* $C(\Omega) > 0$ *such that:*

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} \leq \lambda C(\Omega). \tag{1}$$

Proof: Let us note initially that space V provided the norm $H^1(\Omega)^2$ being a closed subspace of $H^1(\Omega)^2$ is thus an Hilbert space. Let us set

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u}.\nabla \boldsymbol{v}\ d\boldsymbol{x}, \qquad l(\boldsymbol{v}) = \lambda \int_{-1}^{+1} v_1(1,y)\ dy.$$

It is clear, thanks to the Poincaré inequality, that the bilinear continuous form is V-coercive. It is easy to also see that $l \in V'$. One deduces from Lax-Milgram Theorem the existence and uniqueness of $\boldsymbol{u}$ solution of $(\mathcal{S})$. Moreover,

$$\int_\Omega |\nabla \boldsymbol{u}|^2 d\boldsymbol{x} \leq \lambda\sqrt{2}\left(\int_{-1}^{+1} |u_1(1,y)|^2 dy\right)^{1/2},$$

i.e.

$$\|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} \leq \lambda\sqrt{2}\,\|\boldsymbol{u}\|_{L^2(\Gamma)} \leq \lambda\sqrt{2}\,\|\boldsymbol{u}\|_{H^{1/2}(\Gamma)}$$
$$\leq \lambda C_1(\Omega)\,\|\boldsymbol{u}\|_{H^1(\Omega)}$$

Thus there is the estimate (1).□

We now will give an interpretation of the problem $(\mathcal{S})$. One introduces the space

$$\mathcal{V} = \left\{\boldsymbol{v} \in \mathcal{D}(\Omega)^2; \quad \operatorname{div} \boldsymbol{v} = 0\right\}.$$

Let $\boldsymbol{u}$ be the solution of $(\mathcal{S})$. Then, for all $\boldsymbol{v} \in \mathcal{V}$, one has :

$$\langle -\Delta \boldsymbol{u},\, \boldsymbol{v}\rangle_{\mathcal{D}'(\Omega)\times\mathcal{D}(\Omega)} = 0.$$

So that thanks to De Rham Theorem, there exists $p \in \mathcal{D}'(\Omega)$ such that

$$-\Delta \boldsymbol{u} + \nabla p = 0 \text{ in } \Omega. \tag{2}$$

Moreover, since $\nabla p \in H^{-1}(\Omega)^2$, it is known that there exists $q \in L^2(\Omega)$ such that (see [1])

$$\nabla q = \nabla p \quad \text{ in } \Omega. \tag{3}$$

The open Ω being connected, there exists $C \in \mathbb{R}$ such that $p = q + C$, what means that $p \in L^2(\Omega)$. Let us recall that (see [1])

$$\inf_{K \in \mathbb{R}} \|p + K\|_{L^2(\Omega)} \leq C \|\nabla p\|_{H^{-1}(\Omega)^2}.$$

One deduces from the estimate (1) and from (2) that

$$\inf_{K \in \mathbb{R}} \|p + K\|_{L^2(\Omega)} \leq C \|\Delta \boldsymbol{u}\|_{H^{-1}(\Omega)^2} \leq C \|\boldsymbol{u}\|_{H^1(\Omega)^2} \leq \lambda \mathrm{C}(\Omega).$$

Since $\boldsymbol{u} \in H^1(\Omega)^2$ and $\mathbf{0} = -\Delta \boldsymbol{u} + \nabla p \in L^2(\Omega)^2$, it is shown that $-\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n}$ $\in H^{-1/2}(\Gamma)^2$ and one has the Green formula: for all $\boldsymbol{v} \in V$

$$\int_\Omega (-\triangle \boldsymbol{u} + \nabla p) . \boldsymbol{v} \; d\boldsymbol{x} = \int_\Omega \nabla \boldsymbol{u} . \nabla \boldsymbol{v} \; d\boldsymbol{x} + \left\langle -\frac{\partial \boldsymbol{u}}{\partial \mathbf{n}} + p\boldsymbol{n}, \; \boldsymbol{v} \right\rangle, \tag{4}$$

where the bracket represents the duality product $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Moreover, as $p \in L^2(\Omega)$ and $\triangle p = 0$ in Ω, one has $p \in H^{-1/2}(\Gamma)$. Consequently, one has therefore $\dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} \in H^{-1/2}(\Gamma)^2$. The function $\boldsymbol{u}$ being solution of $(\mathcal{S})$, for all $\boldsymbol{v} \in V$ one has:

$$\left\langle \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} - p\boldsymbol{n}, \boldsymbol{v} \right\rangle = \lambda \int_{-1}^{+1} v_1(1, y) \; dy, \tag{5}$$

i.e.

$$\left\langle \frac{\partial \boldsymbol{u}}{\partial x} - p\mathbf{e}_1, \boldsymbol{v} \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial \boldsymbol{u}}{\partial x} + p\mathbf{e}_1, \boldsymbol{v} \right\rangle_{\Gamma_0} = \langle \lambda \mathbf{e}_1, \boldsymbol{v} \rangle_{\Gamma_1}. \tag{6}$$

where $\mathbf{e}_1 = (1, 0)$.

i) Let $\mu \in H_{00}^{1/2}(\Gamma_1)$ and let us set

$$\mu_2 = \begin{cases} \mu \text{ on } \Gamma_0 \cup \Gamma_1 \\ \\ 0 \text{ on } \Gamma_2 \end{cases} \qquad \text{and} \qquad \boldsymbol{\mu} = \begin{pmatrix} 0 \\ \\ \mu_2 \end{pmatrix}$$

where (see [2])

$$H_{00}^{1/2}(\Gamma_1) = \left\{\varphi \in \mathbf{L}^2(\Gamma_1);\, \exists\, \boldsymbol{v} \in \mathbf{H}^1(\Omega), \text{ with } \boldsymbol{v}|_{\Gamma_2} = \mathbf{0},\, \boldsymbol{v}|_{\Gamma_0 \cup \Gamma_1} = \varphi\right\}.$$

It is checked easily that

$$\boldsymbol{\mu} \in\ H^{1/2}(\Gamma)^2 \qquad \text{and} \qquad \int_\Gamma \boldsymbol{\mu}.\boldsymbol{n}\, d\sigma = 0.$$

So that there exists $\boldsymbol{v} \in H^1(\Omega)^2$ satisfying (see [3])

$$\operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \boldsymbol{v} = \boldsymbol{\mu} \quad \text{on} \quad \Gamma.$$

In particular $\boldsymbol{v} \in V$ and according to (6), one has

$$\left\langle \frac{\partial u_2}{\partial x},\, \mu \right\rangle_{\Gamma_1} = \left\langle \frac{\partial u_2}{\partial x},\, \mu \right\rangle_{\Gamma_0},$$

which means that

$$\frac{\partial u_2}{\partial x}|_{\Gamma_1} = \frac{\partial u_2}{\partial x}\,|_{\Gamma_0}\,. \tag{7}$$

One deduces now from (6) that for all $\boldsymbol{v} \in V$,

$$\left\langle \frac{\partial u_1}{\partial x} - p,\, v_1 \right\rangle_{\Gamma_1} + \left\langle -\frac{\partial u_1}{\partial x} + p,\ \ v_1 \right\rangle_{\Gamma_0} = \langle \lambda,\, v_1 \rangle_{\Gamma_1}\,. \tag{8}$$

But, $\operatorname{div} \boldsymbol{u} = 0$ and $u_2|_{\Gamma_1} = u_2\,|_{\Gamma_0}$, one thus has

$$\frac{\partial u_2}{\partial y}|_{\Gamma_1} = \frac{\partial u_2}{\partial y}\,|_{\Gamma_0} \quad \text{and} \quad \frac{\partial u_1}{\partial x}|_{\Gamma_1} = \frac{\partial u_1}{\partial x}\,|_{\Gamma_0}\,. \tag{9}$$

Consequently, thanks to (8) one has:

$$\langle -p,\, v_1 \rangle_{\Gamma_1} + \langle p,\, v_1 \rangle_{\Gamma_0} = \langle \lambda,\, v_1 \rangle_{\Gamma_1} \tag{10}$$

ii) Let $\nu \in H_{00}^{1/2}(\Gamma_1)$ and let us set

$$\nu_1 = \begin{cases} \nu \text{ on } \Gamma_0 \cup \Gamma_1 \\ \\ 0 \text{ on } \Gamma_2 \end{cases} \qquad \text{and} \qquad \boldsymbol{\nu} = \begin{pmatrix} \nu_1 \\ \\ 0 \end{pmatrix}.$$

One easily checks that

$$\boldsymbol{\nu} \in \ H^{1/2}(\Gamma)^2 \qquad \text{and} \qquad \int_\Gamma \boldsymbol{\nu}.\boldsymbol{n} \, d\sigma = 0.$$

So that there exists $\boldsymbol{v} \in H^1(\Omega)^2$ satisfying

$$\text{div } \boldsymbol{v} = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \boldsymbol{v} = \boldsymbol{\nu} \quad \text{on} \quad \Gamma.$$

In particular $\boldsymbol{v} \in V$ and according to (14), one has

$$\langle -p, \nu \rangle_{\Gamma_1} + \langle p, \nu \rangle_{\Gamma_0} = \langle \lambda, \nu \rangle_{\Gamma_1}$$

i.e.

$$p_{|\Gamma_1} = p_{|\Gamma_0} - \lambda \tag{11}$$

where the equality takes place with the $H^{1/2}$ sense. In short, if $\boldsymbol{u} \in H^1(\Omega)^2$ is solution of $(\mathcal{S})$, then there exists $p \in L^2(\Omega)$, unique with an additive constant such that:

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{12}$$

$$\text{div } \boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \tag{13}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_2, \quad \boldsymbol{u}|_{\Gamma_1} = \boldsymbol{u}\,|_{\Gamma_0}, \tag{14}$$

$$\frac{\partial \boldsymbol{u}}{\partial x}|_{\Gamma_1} = \frac{\partial \boldsymbol{u}}{\partial x}\,|_{\Gamma_0}, \tag{15}$$

$$p_{|\Gamma_1} = p_{|\Gamma_0} - \lambda. \tag{16}$$

It is clear that if $(\boldsymbol{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ checks (12)-(16), then $\boldsymbol{u}$ is solution

of $(\mathcal{S})$. Thus it

Theorem 2.2 *The problem (12)-(16) has an unique solution* $(\boldsymbol{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ *up to an additive constant for p. Moreover,* $\boldsymbol{u}$ *verifies* $(\mathcal{S})$ *and*

$$\|\boldsymbol{u}\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)/\,\mathbb{R}} \leq \lambda C(\Omega)\,. \ \square$$

Remark 1 The pressure verifies the relation (16), which means that p satisfies the relation of Patankar et al.[5] .

3 Navier-Stokes Equations

One takes again the assumptions of the Stokes problem given above. For $\lambda \in \mathbb{R}$ given, the one considers the following problem

$$(\mathcal{NS}) \begin{cases} \text{Find } \boldsymbol{u} \in V, \text{ such that} \\ \forall \boldsymbol{v} \in V, \ \displaystyle\int_\Omega \nabla \boldsymbol{u}.\nabla \boldsymbol{v} \ d\boldsymbol{x} + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \ dy \end{cases}$$

with

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Omega (\boldsymbol{u}.\nabla)\,\boldsymbol{v}.\boldsymbol{w} \ d\boldsymbol{x}$$

With an aim of establishing the existence of the solutions of the problem $(\mathcal{NS})$, one uses the Brouwer fixed point theorem (see [4], [6]). One will show it

Theorem 3.1 *The problem* $(\mathcal{NS})$ *has at least a solution* $\boldsymbol{u} \in V$. *Moreover,* $\boldsymbol{u}$ *checks the estimate (1).*

Proof: To show the existence of $\boldsymbol{u}$, one constructs the approximate solutions of the problem $(\mathcal{NS})$ by the Galerkin method and then thanks to the arguments of compactness, one makes a passage to the limit.

i) For each fixed integer $m \geq 1$, one defines an approximate solution $\boldsymbol{u}_m$ of $(\mathcal{NS})$ by

$$\begin{gathered} \boldsymbol{u}_m = \sum_{i=1}^{m} g_{im} \boldsymbol{w}_i, \quad \text{with} \quad g_{im} \in \mathbb{R} \\ ((\boldsymbol{u}_m, \boldsymbol{w}_i)) + b(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{w}_i) = \langle \lambda \boldsymbol{n}, \boldsymbol{w}_i \rangle_{\Gamma_1}, i = 1, ..., m \end{gathered} \tag{17}$$

where $V_m = \langle \boldsymbol{w}_1, ..., \boldsymbol{w}_m \rangle$ vector spaces spanned by the vectors $\boldsymbol{w}_1, ..., \boldsymbol{w}_m$ and $\{\boldsymbol{w}_i\}$ is an Hilbertian basis of V which is separable. Let us note that (17) is equivalent to:

$$\forall \boldsymbol{v} \in V_m, \ ((\boldsymbol{u}_m, \boldsymbol{v})) + b(\boldsymbol{u}_m, \boldsymbol{u}_m, \boldsymbol{v}) = \lambda \int_{-1}^{+1} v_1(1, y) \ dy. \tag{18}$$

With an aim to establish the existence of the solutions of the problem $\boldsymbol{u}_m$, the operator as follows is considered

$$\begin{aligned} \mathrm{P}_m : V_m &\rightarrow \quad V_m \\ \boldsymbol{u} &\longmapsto \mathrm{P}_m(\boldsymbol{u}) \end{aligned}$$

defined by

$$\forall \boldsymbol{u}, \boldsymbol{v} \in V_m, \quad ((\mathrm{P}_m(\boldsymbol{u}), \boldsymbol{v})) = ((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) - \lambda \int_{-1}^{+1} v_1(1, y) \ dy.$$

Let us note initially that P_m is continuous and

$$\forall \boldsymbol{u} \in V, \quad b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}) = 0.$$

Indeed, thanks to the Green formula, one has

$$b(\boldsymbol{u},\, \boldsymbol{u},\, \boldsymbol{u}) = -\tfrac{1}{2}\int_{\Omega} |\boldsymbol{u}|^2 \operatorname{div} \boldsymbol{u}\; d\boldsymbol{x} + \frac{1}{2}\int_{\Gamma} (\boldsymbol{u}.\boldsymbol{n}) \;\; |\boldsymbol{u}|^2\, d\sigma = 0.$$

But, div $\boldsymbol{u} = 0$ in Ω and

$$\int_{\Gamma} (\boldsymbol{u}.\boldsymbol{n})\; |\boldsymbol{u}|^2\, d\sigma = \int_{\Gamma_0} (\boldsymbol{u}.\boldsymbol{n})\; |\boldsymbol{u}|^2\, d\sigma + \int_{\Gamma_1} (\boldsymbol{u}.\boldsymbol{n})\; |\boldsymbol{u}|^2\, d\sigma.$$

since the external normal to Γ_0 is opposed to that of Γ_1 and $\boldsymbol{u} \in V$.

Thanks to Brouwer Theorem, there exists $\boldsymbol{u}_m$ satisfying (18) and

$$\|\boldsymbol{u}_m\|_{\mathbf{H}^1(\Omega)} \leq \lambda C(\Omega)\,. \tag{19}$$

ii) We can extract a subsequence $\boldsymbol{u}_\nu$ such that

$$\boldsymbol{u}_\nu \rightharpoonup \boldsymbol{u} \quad \text{weakly} \quad \text{in} \quad V,$$

and thanks to the compact imbedding of V in $L^2(\Omega)^2$, we obtain

$$\forall \boldsymbol{v} \in V,\; ((\boldsymbol{u}, \boldsymbol{v})) + b(\boldsymbol{u},\, \boldsymbol{u},\, \boldsymbol{v}) = \; \lambda \int_{-1}^{+1} \; v_1(1, y)\;\; dy.$$

As for the Stokes problem, one shows the existence of $p \in L^2(\Omega)$, unique except for an additive constant, such that

$$\begin{cases} -\Delta \boldsymbol{u} + (\boldsymbol{u}.\nabla)\, \boldsymbol{u} + \nabla p = \mathbf{0} & \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{u} = \mathbf{0} & \text{on} \quad \Gamma_2, \\ \boldsymbol{u}|_{\Gamma_1} = \boldsymbol{u}\; |_{\Gamma_0}\,. \end{cases}$$

It is checked finally that

$$\frac{\partial \boldsymbol{u}}{\partial x}|_{\Gamma_1} = \frac{\partial \boldsymbol{u}}{\partial x}\ |_{\Gamma_0} \quad ,$$

$$p_{|\Gamma_1} = p_{|\Gamma_0} - \lambda.$$

Remark 2 i) Theorem 3.1 of problem $(\mathcal{NS})$ *takes place in three dimension.*

ii) One can show that the solution $(\boldsymbol{u}, p)$ *belongs to* $H^2(\Omega)^2 \times H^1(\Omega)$.

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